

Unimodular Random Trees

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Abstract. We consider unimodular random rooted trees (URTs) and invariant forests in Cayley graphs. We show that URTs of bounded degree are the same as the law of the component of the root in an invariant percolation on a regular tree. We use this to give a new proof that URTs are sofic, a result of Elek. We show that ends of invariant forests in the hyperbolic plane converge to ideal boundary points. We also prove that uniform integrability of the degree distribution of a family of finite graphs implies tightness of that family for local convergence, also known as random weak convergence.

§1. Introduction.

The theory of unimodular random rooted networks (URNs) is an outgrowth mainly of two lines of investigations: one is concerned with asymptotics of finite networks, while the other involves group-invariant stochastic processes on infinite Cayley graphs, especially percolation. An important motivation also arises from the class of sofic groups. Parallels with the theory of limits of dense graphs now spur further investigations (see Lovász (2012) for this).

We give full definitions in Section 2, but for now, we recount intuitively some of the above motivations. One way to look at a large finite network (which is a labeled graph) is to look at a large neighborhood around a random uniformly chosen vertex. Often such neighborhood statistics capture quantities of interest and their asymptotics. Thus, one is led to take limits of such statistics and thereby define a probability measure on infinite rooted graphs, where the neighborhood of the root has the statistics that arise as the limit statistics of the finite networks. Such a limit of a sequence of finite networks is called the random weak limit, the local (weak) limit, the distributional limit, or the

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Benjamini-Schramm limit of the sequence. All such limit measures have a property known as unimodularity; it is not known whether all unimodular measures are limits of finite networks. Those that are such limits are called sofic.

Intuitively, a probability measure on rooted networks is unimodular iff its root is chosen “uniformly” from among all its vertices. This, of course, only makes sense for finite graphs. It is formalized for networks on infinite graphs by requiring a sort of conservation property known as the Mass-Transport Principle.

Unimodularity is an extremely powerful property, especially for studying percolation on infinite graphs. In the present context, for example, the component of the identity in a group-invariant percolation on a Cayley graph has a unimodular law as a random graph rooted at the identity.

Consider the following example of a random weak limit of finite graphs: Let \mathbb{T}_3 be a 3-regular tree. Let G_n be the ball of radius n in \mathbb{T}_3 about any point. Since most points in G_n are near the leaves of G_n , the random weak limit of $\langle G_n; n \geq 1 \rangle$ is not \mathbb{T}_3 but the following probability measure, μ . Let T_n be disjoint binary trees of depth n for $n \geq 0$. Modify T_n by adding a new vertex x_n adjacent to the root of T_n . Also consider an isolated vertex x_{-1} . Now add an edge between x_n and x_{n+1} for each $n \geq -1$. The resulting tree, T , was called the *canopy tree* by Aizenman and Warzel (2006). The graph T rooted at x_n is denoted (T, x_n) . We now define μ by letting $\mu(T, x_n) := 2^{-n-2}$ ($n \geq -1$). Thus, μ is supported on a single tree, which is a proper subtree of \mathbb{T}_3 . In fact, as we show, μ can be obtained as the component of the root in an automorphism-invariant percolation on \mathbb{T}_3 . Indeed, one of our main theorems is that every URN that is supported by trees of bounded degree can be obtained as the component of the root in an invariant percolation on a regular tree.

An interesting contrast is provided by other URNs. For example, consider the infinite discrete Sierpiński gaskets characterized by Teplyaev (1998) (see Lemma 2.3 there). These are obtained as the random weak limit of the graphs that are the natural boundaries of the n th-stage construction of the usual Sierpiński gasket as $n \rightarrow \infty$ (see Figure 1). Now, the limit measure μ has an uncountable support, although still all the graphs in the support are subgraphs of the triangular lattice in the plane. Yet in this case, there is no invariant percolation on the triangular lattice such that the component of the root has law μ since the subgraphs in the support of μ have density 0 and by their topology, only one component in any percolation on the triangular lattice can be a Sierpiński gasket (except for degenerate ones that altogether have μ -measure 0).

We call a URN that is supported by trees a URT. We shall use the fact that every URT of bounded degree can be obtained as the law of the component of the root in an

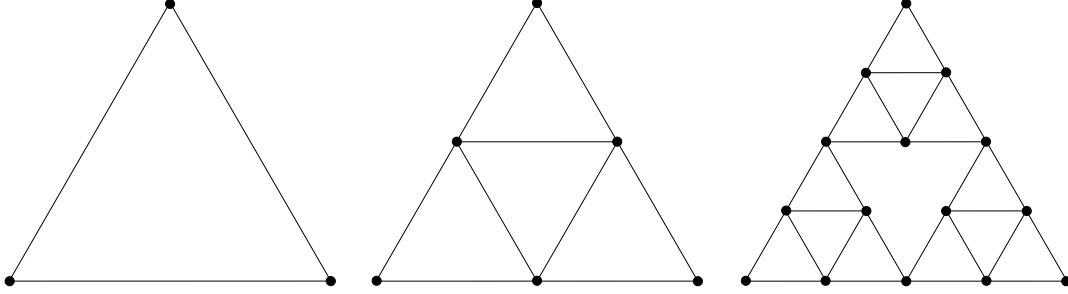


Figure 1. The first three graphs whose random weak limit is the infinite Sierpiński gasket.

invariant labeled percolation on a regular tree to give a new proof that URTs are sofic. This was first shown (in a special case) by Elek (2010), which solved Question 3.3 of Bollobás and Riordan (2011). It was then extended from graphs to networks by Elek and Lippner (2010). Although not needed for any of these results, we give a sufficient condition for a collection of finite graphs to have a convergent subsequence, namely, that their degree distributions be uniformly integrable.

We remark that our theorem showing that every URT of bounded degree can be obtained via invariant percolation on a regular tree has a counterpart in the other direction: That is, rather than put a URT into a regular tree, one can put a regular tree (or forest) on a URT. More precisely, Hjorth (2006) proved that every treeable (probability-measure-preserving) equivalence relation of cost at least 2 can be generated by a free action of a free group \mathbb{F}_2 on 2 generators. If the cost is larger than 1, then there is a subrelation that is generated by a free action of \mathbb{F}_2 : see Proposition 14 of Gaboriau and Lyons (2009). In the remaining case where the treeable equivalence relation has cost 1, the equivalence relation is amenable, whence a theorem of Connes, Feldman, and Weiss (1981) shows that it is generated by a free action of \mathbb{Z} . A URT is essentially the same as a treeable equivalence relation. We do not use any of these notions here, so leave these terms undefined.

Finally, consider a discrete forest in the hyperbolic plane. A simple infinite path in the forest is called a ray. Clearly a ray can have fairly arbitrary limiting behavior; in particular, though it must tend to the ideal boundary because the forest is discrete, the ray need not converge to any ideal boundary point. However, we show that with the condition solely that the forest is random with a law that is invariant under hyperbolic isometries, a.s. all its rays converge to ideal boundary points. We do not know whether this holds in higher dimensions or, more generally, in word-hyperbolic groups. We also do not know whether all rays converge with positive speed.

§2. Definitions.

We review a few definitions from the theory of unimodular random rooted networks; for more details, see Aldous and Lyons (2007). A **network** is a (multi-)graph $G = (V, E)$ together with a complete separable metric space Ξ called the **mark space** and maps from V and E to Ξ . Images in Ξ are called **marks**. Each edge is given two marks, one associated to (“at”) each of its endpoints. The only assumption on degrees is that they are finite. We omit the mark maps from our notation for networks.

A **rooted network** (G, o) is a network G with a distinguished vertex o of G , called the **root**. A **rooted isomorphism** of rooted networks is an isomorphism of the underlying networks that takes the root of one to the root of the other. We do not distinguish between a rooted network and its isomorphism class. Let \mathcal{G}_* denote the set of rooted isomorphism classes of rooted *connected* locally finite networks. Define a separable complete metric on \mathcal{G}_* by letting the distance between (G_1, o_1) and (G_2, o_2) be $1/(1 + \alpha)$, where α is the supremum of those $r > 0$ such that there is some rooted isomorphism of the balls of (graph-distance) radius $\lfloor r \rfloor$ around the roots of G_i such that each pair of corresponding marks has distance less than $1/r$. For probability measures μ, μ_n on \mathcal{G}_* , we write $\mu_n \Rightarrow \mu$ when μ_n converges weakly with respect to this metric.

For a (possibly disconnected) network G and a vertex $x \in V(G)$, write G_x for the connected component of x in G . If G is finite, then write U_G for a uniform random vertex of G and $U(G)$ for the corresponding distribution of (G_{U_G}, U_G) on \mathcal{G}_* . Suppose that G_n are finite networks and that μ is a probability measure on \mathcal{G}_* . We say that the **random weak limit** of G_n is μ if $U(G_n) \Rightarrow \mu$.

A probability measure that is a random weak limit of finite networks is called **sofic**. In particular, a group is called sofic when its Cayley diagram is sofic. All sofic measures are unimodular, which we now define. Similarly to the space \mathcal{G}_* , we define the space \mathcal{G}_{**} of isomorphism classes of locally finite connected networks with an ordered pair of distinguished vertices and the natural topology thereon. We shall write a function f on \mathcal{G}_{**} as $f(G, x, y)$. We refer to $f(G, x, y)$ as the **mass** sent from x to y in G .

DEFINITION 2.1. Let μ be a probability measure on \mathcal{G}_* . We call μ **unimodular** if it obeys the **Mass-Transport Principle**: For all Borel $f : \mathcal{G}_{**} \rightarrow [0, \infty]$, we have

$$\int \sum_{x \in V(G)} f(G, o, x) d\mu(G, o) = \int \sum_{x \in V(G)} f(G, x, o) d\mu(G, o). \quad (2.1)$$

It is easy to see that every sofic measure is unimodular, as observed by Benjamini and Schramm (2001b), who introduced this general form of the Mass-Transport Principle

under the name “intrinsic Mass-Transport Principle”. The converse is open and was posed as a question by Aldous and Lyons (2007).

A special form of the Mass-Transport Principle was considered, in different language, by Aldous and Steele (2004). Namely, they defined μ to be *involution invariant* if (2.1) holds for those f supported on (G, x, y) with $x \sim y$. In fact, the Mass-Transport Principle holds for general f if it holds for these special f , as shown by Aldous and Lyons (2007):

PROPOSITION 2.2. *A measure is involution invariant iff it is unimodular.*

See also Benjamini and Curien (2010) for a discussion of unimodularity.

We call a measure a **URT** if it is a unimodular probability measure on rooted networks whose underlying graphs are trees. We call a probability measure a *labeled percolation* on a graph G if it is carried by the set of networks on G whose marks are pairs, with the second coordinate, called *color*, of a mark being 0 or 1. Edges colored 0 or 1 are called *closed* and *open*, respectively.

§3. Tightness and Degree.

One of our theorems is that URTs are sofic. Although for this purpose, we shall not need results that imply random weak convergence of a subsequence of finite graphs, such results have not been given in the literature before except in the easy case of bounded degree. On the other hand, it does not suffice that the mean degrees be bounded. For example, consider the complete bipartite graphs $K_{2,n}$: No subsequence converges. Yet also, it is not necessary that the mean degrees be bounded. In fact, the mean degree can be infinite for extremal sofic unimodular random rooted graphs, even trees. Here, *extremal* means that the probability measure is not a convex combination of other unimodular probability measures on rooted graphs. We impose that condition since it is trivial to take a mixture of finite-mean-degree URTs to get a URT of infinite mean degree.

For examples, let $\langle p_n ; n \geq 1 \rangle$ be a probability distribution on \mathbb{Z}^+ with infinite mean. For each integer k , join k to $k + 1$ by n parallel edges with probability p_n , independently for different k . This is easily seen to be an extremal sofic probability measure. To get an extremal URT with infinite mean degree, take the universal cover rooted at 0 of the resulting multigraph; see Example 9.3 of Aldous and Lyons (2007).

Thus, it may be useful to present the following result on tightness. For simplicity and with no essential loss of generality, we shall assume that all our graphs have no isolated vertices.

THEOREM 3.1. *If A is a family of finite graphs such that the random variables $\{\deg_G U_G; G \in A\}$ are uniformly integrable, then $\{U(G); G \in A\}$ is tight.*

Proof. Let $f(d) := \sup_{G \in A} \mathbf{E}[\deg_G U_G; \deg_G U_G > d]$. By assumption, $\lim_{d \rightarrow \infty} f(d) = 0$. Write $m(G) := \mathbf{E}[\deg_G U_G]$. Thus, $1 \leq m(G) \leq f(0) < \infty$. Write $D(G)$ for the degree-biased probability measure on $\{(G, x); x \in V(G)\}$, that is,

$$D(G)[(G, x)] = \frac{\deg_G x}{m(G)} \cdot U(G)[(G, x)],$$

and D_G for the corresponding root. Since $U(G) \leq m(G)D(G) \leq f(0)D(G)$, it suffices to show that $\{D(G); G \in A\}$ is tight.

Now $f(d) = \sup_{G \in A} \mathbf{P}[\deg_G D_G > d] \cdot m(G)$. Fix $\epsilon > 0$. Choose d_0 so that $f(d_0) < \epsilon/2$. Then choose recursively $\langle d_r; r \geq 1 \rangle$ so that $d_{r+1} \geq d_r$ and

$$f(d_{r+1}) < \frac{\epsilon}{2^{r+2} \prod_{i=0}^r d_i}.$$

We claim that for all $G \in A$,

$$\mathbf{P}[|B_r(D_G)| > d_r^r] < \epsilon, \quad (3.1)$$

where $B_r(x)$ denotes the vertices in G within distance r of x . Note that (3.1) implies that $\{D(G); G \in A\}$ is tight.

Fix $G \in A$. Write $S_r(x)$ for the set of vertices in G at distance exactly r from x . We shall prove by induction on r that

$$\mathbf{P}[\exists i \in [0, r] \exists x \in S_i(D_G) \deg_G x > d_i] < (1 - 2^{-(r+1)})\epsilon. \quad (3.2)$$

First, $\mathbf{P}[\deg_G D_G > d_0] \leq f(d_0)/m(G) < \epsilon/2$, which is the case $r = 0$ of (3.2). Let $\langle X_n; n \geq 0 \rangle$ be simple random walk on G starting at $X_0 = D_G$. Then (G, X_n) has the same law as $D(G)$ by involution invariance. Now suppose that (3.2) holds for $r = t$ and let us prove it for $r = t+1$. Let W be the set of $y \in V(G)$ such that for all $i \in [0, t]$ and for all $x \in S_i(y)$, we have $\deg_G x \leq d_i$. For $y \in W$, we have that $|S_{t+1}(y)| \leq \prod_{i=0}^t d_i$. Since

$$\mathbf{P}[\deg_G X_{t+1} > d_{t+1}] \leq f(d_{t+1})/m(G) \leq \frac{\epsilon}{2^{t+2} \prod_{i=0}^t d_i},$$

it follows that

$$\mathbf{P}[D_G \notin W \text{ or } \exists x \in S_{t+1}(D_G) \deg_G x > d_{t+1}] < (1 - 2^{-(t+1)})\epsilon + \epsilon/2^{t+2} = (1 - 2^{-(t+2)})\epsilon,$$

as desired. This proves (3.2).

It follows that

$$\mathbf{P}[\exists x \in B_r(D_G) \deg_G x > d_r] < \epsilon,$$

which implies (3.1). ■

In this proof, the only way that we used finiteness of the graphs was that the degree-biased uniform distribution on vertices gave a stationary measure for simple random walk. Thus, the result applies also to any collection of probability measures bounded by a fixed multiple of stationary probability measures on rooted graphs, such as unimodular probability measures on graphs.

§4. Invariant Percolation.

We now prove that every URT of bounded degree arises as the open component of the root in an invariant percolation on a regular tree.

We use the following lemma that is straightforward to check from the definitions.

LEMMA 4.1. *Suppose that μ is a unimodular probability measure on rooted networks. Let ϕ be a measurable map on rooted networks that takes each network to an element of the mark space. Define Φ to be the map on rooted networks that takes a network (G, o) to another network on the same underlying graph, but replaces the mark at each vertex $x \in G$ by $\phi(G, x)$. Then the pushforward measure $\Phi_*\mu$ is also unimodular. If instead we add a second coordinate to each vertex mark by an IID mark, then the resulting measure is again unimodular.*

THEOREM 4.2. *Let μ be a probability measure on rooted networks whose underlying graphs are trees of degree at most d . Then μ is unimodular iff μ is the law of the open component of the root in a labeled percolation on a d -regular tree whose law is invariant under all automorphisms of the tree.*

Proof. The “if” part of the assertion is well known and not dependent on the fact that the underlying graph is a tree. See, e.g., Benjamini, Lyons, Peres, and Schramm (1999) or Theorem 3.2 of Aldous and Lyons (2007).

The idea for proving the converse is as follows. First sample $(T, o) \sim \mu$. Of all the possible ways to embed it in the d -regular tree \mathbb{T}_d such that o maps to the root \mathbf{o} of \mathbb{T}_d , choose one uniformly (i.e., choose one arbitrarily and then apply a uniform automorphism of \mathbb{T}_d preserving \mathbf{o}). The embedded image of T is marked open. Now for every edge e in \mathbb{T}_d that is not in the image of T but has one endpoint in T , mark e closed and sample an independent copy $(T', o') \sim \mu$ with o' embedding as the endpoint of e that is not in T . However, this choice of T' has to be biased so that the degree of o' is not d . In fact, we sample instead $(T', o') \sim \mu'$, where μ' is absolutely continuous with respect to μ with Radon-Nikodým derivative at (T, o) equal to $(d - \deg_T o)/\alpha$, where α is a normalizing constant. Continue in this way to cover all of the vertices of \mathbb{T}_d by weighted independent

copies of $(T', o') \sim \mu'$. Of course, all edges in embedded copies of T or T' are marked open, while the rest are marked closed. To prove that this is invariant, we first show involution invariance of the constructed marked tree and then appeal to Theorem 3.2 of Aldous and Lyons (2007) to get that it is an invariant percolation on \mathbb{T}_d . Proving involution invariance involves two cases: one case involves crossing a closed edge; that's where the biased measure μ' comes in. The other case involves crossing an open edge; that's where the unimodularity of the original measure μ comes in.

Here are the details. Let p_k be the μ -probability that the root has k children. Let $\alpha := \sum_k p_k(d - k)$ be a normalizing constant. Let μ' be absolutely continuous with respect to μ with Radon-Nikodým derivative at (T, o) equal to $(d - \deg_T o)/\alpha$. Given two probability measures ν_1 and ν_2 supported by networks on rooted trees where the root has degree at most $d - 1$ and all other vertices have degree at most d , write $Q(\nu_1, \nu_2)$ for the probability measure supported by networks on the rooted d -ary tree constructed as follows, similar to a Galton-Watson branching process: Choose $(T', o) \sim \nu_1$, whose edges are colored open. To each vertex $x \neq o$ of T' , adjoin $d - \deg_{T'} x$ edges colored closed whose other endpoint is the root of an independent sample from ν_2 , while to the root o of T' , adjoin $d - 1 - \deg_{T'} o$ edges colored closed whose other endpoint is the root of an independent sample from ν_2 . Call the result the network (T, o) . Then $Q(\nu_1, \nu_2)$ is the law of (T, o) . Write ν for the measure on rooted networks defined by the equation $\nu = Q(\mu', \nu)$.

Let ρ be the measure constructed as follows: Choose $(T', o) \sim \mu$, whose edges are colored open. To each vertex x of T' , adjoin $d - \deg_{T'} x$ edges colored closed whose other endpoint is the root of an independent sample from ν . The result is a network whose underlying graph is \mathbb{T}_d . We claim that this measure ρ is unimodular, which we show by proving that ρ is involution invariant. This suffices to prove the theorem by appeal to Theorem 3.2 of Aldous and Lyons (2007) (in which the averaging over automorphisms is taken).

To prove this claim, it will be convenient to use the following technical modification of ρ to deal with counting issues: Given $(T, o) \sim \rho$, assign independently and uniformly marks to the closed edges in each direction so that each vertex is surrounded by outgoing closed edges marked $1, \dots, k$ when it is incident to k closed edges. Call ρ' the resulting measure on networks. It clearly suffices to prove that ρ' is involution invariant.

For k such that $p_k > 0$, let μ_k be the measure constructed as follows: Choose $(T', o) \sim \mu$, whose edges are colored open, conditioned on $\deg_{T'} o = k$. To each vertex $x \neq o$ of T' , adjoin $d - \deg_{T'} x$ edges colored closed whose other endpoint is the root of an independent sample from ν . Let \mathcal{N}_i denote the class of networks supported on a rooted tree with all vertices except the root having degree d and the root having degree i . Consider $i, i' \in$

$[1, d-1]$ and Borel sets $A \subseteq \mathcal{N}_i$, $A' \subseteq \mathcal{N}_{i'}$, and $B_1, \dots, B_{d-i-1}, B'_1, \dots, B'_{d-i'-1} \subseteq \mathcal{N}_{d-1}$. Now let $(T, o) \sim \rho'$ and let o' be a uniform neighbor of o . Then the chance that we see i open edges at o , the edge (o, o') is closed with mark j in the direction (o, o') and mark j' in direction (o', o) , see i' open edges at o' , and see the event where the open edges at o are part of a network in A , the open edges at o' are part of a network in A' , while the other endpoints of the closed edges at o belong to networks in (B_1, \dots, B_{d-i-1}) in increasing order of their marks from o and similarly the other endpoints of the closed edges at o' belong to networks in $(B'_1, \dots, B'_{d-i'-1})$ in increasing order of their marks from o' (see Figure 2) equals

$$p_i \mu_i(A) \prod_{m=1}^{d-i-1} \nu(B_m) \cdot \frac{d-i}{d} \cdot \frac{1}{(d-i)(d-i')} \cdot \prod_{r=1}^{d-i'-1} \nu(B'_r) \cdot p_{i'} \frac{d-i'}{\alpha} \mu_{i'}(A').$$

This is invariant under the involution exchanging o and o' .

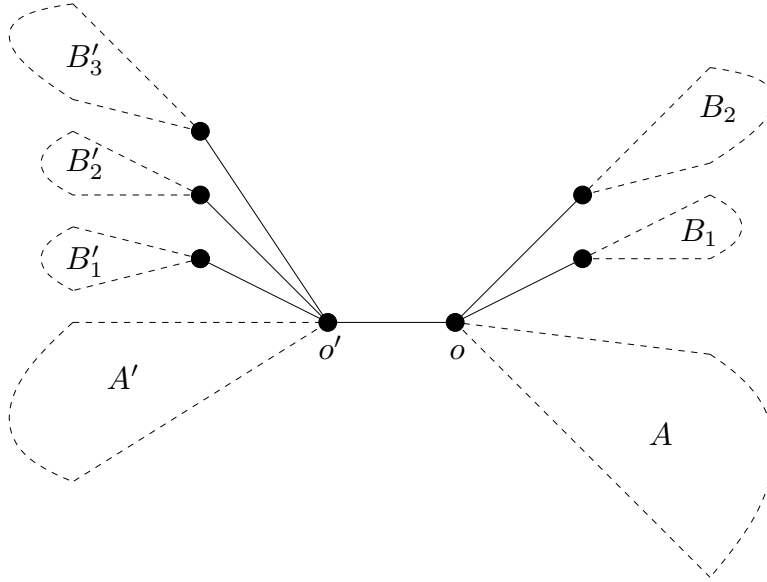


Figure 2. All edges drawn with solid lines are closed. The root o is incident to 3 closed edges, while o' is incident to 4 closed edges.

The other case to prove is when the edge (o, o') is open. Consider the following measure, σ . Begin with a sample $(T, o) \sim \mu$. Assign a second coordinate $(\tau_1(x), \dots, \tau_{d-1}(x))$ to the vertex mark at each vertex x given by IID samples $\tau_i(x) \sim \nu$. This new network is unimodular by Lemma 4.1. Now replace the second coordinate of the vertex mark at each vertex x by $(\tau_1(x), \dots, \tau_{d-\deg_T(x)}(x))$. This new network is again unimodular by Lemma 4.1. We denote by σ its law. Note that we can obtain ρ from σ by replacing the

second coordinate of the vertex mark at each x by a tree network rooted at x , where we adjoin $d - \deg_T(x)$ closed edges to x , at the other end of which we adjoin the trees $\tau_i(x)$.

What remains to prove is that involution invariance holds for ρ' across open edges. It suffices to do the same for ρ . But this is clearly the same as unimodularity of σ . ■

§5. Soficity.

We now use the preceding theorem to prove that URTs are sofic. This result is the same as Theorem 4 of Elek and Lippner (2010), but in different language. See Example 9.9 of Aldous and Lyons (2007) for a comparison of the different languages.

THEOREM 5.1. *Every URT is sofic.*

Proof. It follows from Theorem 3.4 of Bowen (2003) that every invariant network on \mathbb{T}_d is sofic; the result is stated there for even d only, but the proof works for all d . Thus, given a URT μ , if the degrees are bounded by d , let ρ be an invariant labeled percolation on \mathbb{T}_d such that the open component of the root has law μ . Let $\langle G_n; n \geq 0 \rangle$ be finite networks whose random weak limit is ρ . Here, we may assume that the edges of G_n are each colored closed or open. Let G'_n be the result of deleting every closed edge from G_n . Then clearly $\langle G'_n \rangle$ has random weak limit μ . Finally, if the degrees are not bounded μ -a.s., then for each d , let μ_d be the law of the component of o when we delete every edge of T incident to some vertex of degree larger than d , where $(T, o) \sim \mu$. Then μ_d is unimodular and, by what we just proved, sofic. Since the sofic measures form a weakly closed set and μ_d tend weakly to μ , we deduce that μ is sofic as well. ■

As noted by Elek and Lippner (2010), this implies that every treeable group is sofic. Here, a group Γ is **treeable** if there is a probability measure on trees with vertex set Γ that is invariant under the natural action of Γ ; such a probability measure is called a **treeing** of Γ . Briefly, the idea is to use a treeing μ of Γ , a generating set S for Γ , and a sofic approximation $\langle G_n \rangle$ of μ to construct a sofic approximation of the Cayley diagram of Γ with respect to S by putting edges labeled $s \in S$ between points x, y of G_n such that a path from x to y has length at most R_n and has labels that multiply to s , where $R_n \rightarrow \infty$ at an appropriately slow rate.

§6. Rays.

Random weak limits of finite trees have mean degree at most 2, are supported by trees with at most 2 ends, and hence are recurrent for simple random walk; see Proposition 6.3 of Aldous and Lyons (2007). In the case of URTs with finite mean degree larger than 2, the speed of simple random walk is positive: see Theorem 4.9 of Aldous and Lyons (2007). The case of infinite mean degree is open. However, it is interesting in all cases to see whether the rays themselves, rather than simple random walk, have positive speed when embedded in a larger graph. What we mean by this is the following.

We say that a sequence $\langle x_n; n \geq 0 \rangle$ in a metric space has **positive (liminf) speed** if there is some constant $c > 0$ such that the distance between x_n and x_0 is at least cn for all $n \geq 1$. A simple infinite path in a tree is called a **ray**. An **end** of a tree is an equivalence class of rays, where two rays are equivalent when they have finite symmetric difference. Of course, any statement about limits of rays applies equally to all rays belonging to the same end and is therefore a statement about limits of ends.

We are interested in the rays in forests that arise either as invariant percolation on a Cayley graph or as random graphs discretely embedded in hyperbolic space \mathcal{H}^d with an isometry-invariant law. When do all the rays have positive speed in the metric of the Cayley graph or in the hyperbolic metric? It does not suffice that the Cayley graph be non-amenable; e.g., consider the usual Cayley graph of the group $\mathbb{Z} * \mathbb{Z}^2$. The random forest we use arises from an independent copy of the uniform spanning tree (Pemantle, 1991) in every copy of \mathbb{Z}^2 . Then almost surely, each such tree contains only one end and no ray has positive speed. (In fact, the n th vertex in a ray is roughly at distance $n^{4/5}$ from its starting point; see Barlow and Masson (2010).) Thus, we must strengthen the assumption on the Cayley graph if we want the conclusion to hold.

QUESTION 6.1. Does every ray in an invariant forest have positive speed a.s. in a word-hyperbolic group?

We do not know the answer. Thus, we consider the following still weaker property:

QUESTION 6.2. Does every ray in an invariant forest in a word-hyperbolic group a.s. converge to an ideal boundary point?

See the survey Kapovich and Benakli (2002) for information on the boundary of a word-hyperbolic group. We know the answer only in \mathcal{H}^d for $d = 2$:

THEOREM 6.3. *Every ray in an invariant forest in a one-ended graph G embedded in \mathcal{H}^2 such that the isometry group of \mathcal{H}^2 acts quasi-transitively on G a.s. converges to an ideal boundary point.*

Here, to say that G is one-ended means that the complement of each finite set in G has only one infinite component in G . Note that the ideal boundary points of G are the same as those of \mathcal{H}^2 .

One can prove a similar statement for forests in \mathcal{H}^2 whose law is invariant under isometries of \mathcal{H}^2 . However, we do not know whether all rays have positive speed, even in \mathcal{H}^2 .

Proof. We may assume that the forest \mathfrak{F} is spanning and contains only infinite trees without loss of generality. Indeed, we may first delete all finite trees and then independently add an edge at random from each vertex not in the forest to a vertex that is closer to the forest.

Let G^\dagger be the planar dual graph of G ; its edges are in bijective correspondence with those of G in such a way that each edge e of G crosses only its corresponding edge e^\dagger . Let \mathfrak{F}^\times be the dual spanning forest in G^\dagger defined by $e^\dagger \in \mathfrak{F}^\times$ iff $e \notin \mathfrak{F}$. Now G and G^\dagger are unimodular since the isometry group of \mathcal{H}^2 is unimodular.

If every tree of \mathfrak{F} has at least 3 ends, then simple random walk on the forest \mathfrak{F} a.s. has positive speed in the metric of the forest (as we noted above or, e.g., by Theorem 16.4 of Lyons with Peres (2012)), whence it also has positive speed in the graph metric of G by Lemma 4.6 of Benjamini, Lyons, and Schramm (1999), and hence also in the hyperbolic metric. Therefore, as in the proof of Theorem 4.1 of Benjamini and Schramm (2001a), simple random walk on the forest \mathfrak{F} a.s. converges to an ideal boundary point. It follows that a.s. for every tree T of the forest, μ_T -a.e. ray converges to an ideal point, where μ_T is harmonic measure on the boundary of T . By the Mass-Transport Principle, a.s. for every tree T of \mathfrak{F} , μ_T has full support in the boundary of T : if not, then there are maximal rooted subtrees (T', x) of T whose boundary has harmonic measure 0. Let each vertex in T' send mass 1 to x in such a situation. Then x receives infinite mass, so by the Mass-Transport Principle, this has probability 0, as desired. By planarity, this means that a.s. every ray, not merely μ_T -a.e. ray, converges to an ideal point.

Now if only some trees in \mathfrak{F} have at least 3 ends, then what we have just proved applies to those trees since we may delete the other trees and then add edges as in the first paragraph to get a spanning forest all of whose trees have at least 3 ends. Of course, the same applies to \mathfrak{F}^\times .

If no trees have two ends, then the fact that all trees in \mathfrak{F} or in \mathfrak{F}^\times with at least 3 ends have all their rays convergent implies that all rays in the 1-ended trees converge as well.

Finally, it remains to deal with the case that some trees have 2 ends. The following construction will be useful. Given 3 distinct ideal boundary points ξ_1 , ξ_2 , and ξ_3 , let Δ be

the ideal triangle in \mathcal{H}^2 whose vertices are those boundary points. Let z be the center of Δ and $Z(\xi_1, \xi_2, \xi_3)$ be the set of vertices of G that are closest in \mathcal{H}^2 to z .

Suppose first that some tree in \mathfrak{F} or \mathfrak{F}^\times has 1 or at least 3 ends. Then there cannot be only one common limit point of all the rays belonging to those trees since one cannot pick an ideal boundary point in an invariant way (i.e., there is no probability measure on the ideal boundary that is invariant; this fact holds for every non-elementary word-hyperbolic group—see Theorems 4.3 and 2.28 of Kapovich and Benakli (2002)). Therefore, a.s. no end of a 2-ended tree can have as its limit points the entire ideal boundary. If a particular end of a tree T does not converge, then its set of limit points is a proper arc. In this case, let $A(T)$ be the set of 3 or 4 endpoints of the arcs of limit points of the 2 ends of T . Then let $X(T)$ be the union of $Z(B)$ over all subsets of $A(T)$ of size 3, where $Z(\cdot)$ is as defined above. Now transport mass $1/|X(T)|$ from each vertex of T to each vertex of $X(T)$. Since $X(T)$ is finite and yet each point of $X(T)$ receives infinite mass, this has probability 0 by the Mass-Transport Principle.

On the other hand, we claim that a.s. not all trees in \mathfrak{F} and \mathfrak{F}^\times have 2 ends. For when they do, we can order the trees as the integers in the following sense. Each tree T in \mathfrak{F} separates the plane into two pieces since it has two ends. The dual of the edge boundary of T lies in \mathfrak{F}^\times and has two connected components, each one being part of a tree in \mathfrak{F}^\times . The same applies to each of those trees in turn, which means that on each side of those trees, besides T , there is another tree in \mathfrak{F} that includes the dual of part of its edge boundary. Those two trees in \mathfrak{F} are the ones next to T in the integer ordering of all the trees in \mathfrak{F} . This allows us to define an invariant percolation with all clusters finite yet with arbitrarily high marginal, contradicting non-amenability by Theorem 2.12 of Benjamini, Lyons, Peres, and Schramm (1999). To see this, call the unique bi-infinite simple path in a tree with 2 ends the **trunk** of that tree. If a vertex x of the trunk is deleted and y is in a finite component of what is left of the tree (or $y = x$), then call x the **trunk attachment** of y . Now given $\epsilon > 0$, delete each tree with probability ϵ independently and in each tree that is left, delete each vertex on the trunk with probability ϵ independently, and delete all vertices not on a trunk whose trunk attachment was deleted. Thus, each vertex is deleted with probability $\epsilon + (1 - \epsilon)\epsilon$. It remains to show that the graph induced by the remaining vertices has no infinite clusters a.s. Number the trees by \mathbb{Z} as indicated above, where we choose arbitrarily which tree is numbered 0 and in which direction the integers increase. Suppose that trees numbered m and $m + n + 1$ are deleted, while the n trees numbered i are not deleted for $m < i < m + n + 1$. Consider a vertex x_1 on the trunk of tree number $m + 1$. Then there is at least one vertex x_2 on the trunk of tree number $m + 2$ such that for some y_1 whose trunk attachment is x_1 and some y_2 whose

trunk attachment is x_2 , there is an edge of G between y_1 and y_2 . Likewise, we may choose x_3 on the trunk of the tree number $m+3$ such that some y_3 whose trunk attachment is x_3 is adjacent to y_2 , etc. If all vertices x_1, \dots, x_n are deleted, then there is a path of deleted vertices stretching from tree number m to tree number $m+n+1$. This has probability ϵ^n . We may choose infinitely many such sequences $\langle x_1, \dots, x_n \rangle$ that are pairwise disjoint, so that the corresponding events that these sequences are deleted are independent, each having the same probability ϵ^n . Therefore, infinitely many such events occur a.s., and when they do, they separate the remaining vertices between trees m and $m+n+1$ into finite components. Since this happens between each consecutive pair of deleted trees, all components are finite a.s. This completes the proof of the theorem. \blacksquare

We can give some additional information about the limit points of the rays in those trees with at least 3 ends. We call the set of limit points of the rays in a tree the **limit set** of that tree.

THEOREM 6.4. *Let G be a one-ended graph embedded in \mathcal{H}^2 such that the isometry group of \mathcal{H}^2 acts quasi-transitively on G . Let \mathfrak{F} be an invariant forest in G such that each tree has at least 3 ends. If \mathfrak{F} has only one tree, then a.s. its limit set is the entire ideal boundary. Otherwise, a.s. for each tree: the limit set is a perfect nowhere-dense set and the map from ends to limit points never maps more than 2 ends to the same limit.*

Proof. Suppose first that \mathfrak{F} is a single tree whose limit set is not the entire ideal boundary. Since the limit set is closed, its complement contains a non-empty arc. For every such maximal arc, I , let P be the geodesic whose limit points are the endpoints of I . Consider a Poisson point process on P of unit intensity, from each point of which we start a geodesic perpendicular to P towards I . By divergence of geodesics, the region R between any consecutive pair of such geodesics contains infinitely many vertices of G . Let each vertex of G in R send mass 1 total, split equally among the vertices of \mathfrak{F} that are closest to (or in) R (of which there are only finitely many). By the Mass-Transport Principle, this has probability 0, as desired.

Now suppose that \mathfrak{F} has more than one tree. Let T be one of them. If its limit set is not nowhere dense, then it contains a maximal proper arc. Let I be one such arc. Among all the bi-infinite paths in T both of whose ends converge to points in I , there is only one, call it P , such that all others lie on the same side of P as I . Then each end with a limit in I contains a unique ray that starts at a point in P . Let each vertex in such a ray send mass 1 to its starting point in P . Then some points in P get infinite mass, so by the Mass-Transport Principle, this has probability 0. This establishes that the limit set is a.s. nowhere dense. Similarly, the map from ends to limit points a.s. never maps more than

2 ends to the same limit. In particular, the limit set is a.s. infinite. If it is not perfect, then there is an isolated limit point, ξ , and a vertex x of T such that three rays from x that are disjoint other than at x have distinct limit points, one being ξ . In fact, for each isolated limit point ξ , there is a unique such x that is “closest” to ξ in that the ray from x to ξ contains no other vertex with these properties. But then we can transport to x mass 1 from each vertex on the ray from x to ξ and so, by the Mass-Transport Principle, this has probability 0. ■

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